

OCTAHEDRAL NORMS IN TENSOR PRODUCTS OF BANACH SPACES

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ABSTRACT. We continue the investigation of the behaviour of octahedral norms in tensor products of Banach spaces. Firstly, we will prove the existence of a Banach space Y such that the injective tensor products $l_1 \widehat{\otimes}_\varepsilon Y$ and $L_1 \widehat{\otimes}_\varepsilon Y$ both fail to have an octahedral norm, which solves two open problems from the literature. Secondly, we will show that in the presence of the metric approximation property octahedrality is preserved from a non-reflexive L -embedded Banach space taking projective tensor products with an arbitrary Banach space.

1. INTRODUCTION

According to [12, Remark II.5.2], the norm of a Banach space X is *octahedral* if, for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there exists $y \in S_X$ such that

$$\|x + \lambda y\| \geq (1 - \varepsilon)(\|x\| + |\lambda|) \text{ for every } x \in E \text{ and every } \lambda \in \mathbb{R}.$$

The starting point of dual characterisations of octahedral norms was in [9], where the author proved that if a Banach space X has an octahedral norm then the dual X^* enjoys the *weak* strong diameter two property* (w^* -SD2P), i.e. every convex combination of weak-star slices of the dual unit ball has diameter two. The converse of this result was proved in [6, Theorem 2.1] (see also [14, 20]). It follows that a Banach space has the *strong diameter two property* (SD2P) (i.e. every convex combination of slices of the unit ball has diameter two) if, and only if, the dual norm is octahedral. This characterisation motivated a lot of research on octahedral norms in connection with the so called “big slice phenomenon” and it will be used repeatedly without reference throughout this text.

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The connection between the SD2P and octahedrality was the basis for new results related to the big slice phenomenon in tensor product spaces and, by duality, in spaces of operators. Indeed, in [7, Theorem 2.5] it was proved that given two Banach spaces X and Y such that the norms of X^* and Y are octahedral then the norm of every closed subspace H of $L(X, Y)$ which contains finite-rank operators is octahedral. As a corollary, the projective tensor product of two spaces having the SD2P enjoys the SD2P, a result which improved the main results of [4] and gave a partial answer to [3, Question (b)], where it was asked how diameter two properties are preserved by tensor product spaces. However, it remained an open problem whether the assumption of the SD2P on one of the factor can be eliminated [7, p. 177]. In [21, Theorem 2.2] a result similar to [7, Theorem 2.5] appeared, proving that octahedrality is preserved by taking injective tensor products from both factors. But the question whether the assumption on one of the factor can be removed remained open [21, Question 4.1] (see also [15, p. 5]).

Dually, it is a natural question how octahedrality is preserved by projective tensor products. There are several examples [21, Examples] which suggest that it should be sufficient to assume octahedrality on one of the factors for the projective tensor product to have an octahedral norm, and this was posed as an open problem [21, Question 4.4]. Even the particular case of Lipschitz-free spaces have been considered [8, Question 2].

The aim of this note is to continue studying octahedrality in tensor product spaces and to give some complete and some partial answers to the above questions. We start by giving definitions and preliminary results in Section 2. In Section 3 we will prove that there are Banach spaces Y such that the injective tensor products $\ell_1 \widehat{\otimes}_\varepsilon Y$ and $L_1 \widehat{\otimes}_\varepsilon Y$ fail to have an octahedral norm. Indeed, we will characterise in Theorem 3.10 when the spaces $X \widehat{\otimes}_\varepsilon Y$ have an octahedral norm whenever X is either ℓ_1 or L_1 and Y is either ℓ_p or ℓ_p^n for $1 \leq p \leq \infty$ and $n \geq 2$. This will give a negative answer to [21, Question 4.1] and to a question from [7, p. 177]. Moreover, Theorem 3.10 also gives a complete answer to the problem of how the SD2P is preserved by projective tensor products, posed in [3, Question (b)]. In Section 4 we study octahedrality of projective tensor products. In Theorem 4.3 we will prove that octahedrality is preserved from one of the factors by taking projective tensor products in presence of the metric approximation property whenever one of the factors in a non-reflexive L -embedded Banach space, which provides a partial positive answer to [21, Question 4.4].

2. NOTATION AND PRELIMINARIES

We will only consider real and non-zero Banach spaces and we follow standard Banach space notation as used in e.g. [5]. Given a Banach space X we denote the closed unit ball by B_X and the closed unit sphere by S_X . The Banach space of bounded linear operators from X to a Banach space Y is denoted by $L(X, Y)$, while the subspace of finite rank operators is denoted by $F(X, Y)$. By L_1 we mean the Banach space $L_1[0, 1]$. By p^* we denote the conjugate exponent of $1 \leq p \leq \infty$ defined by $\frac{1}{p} + \frac{1}{p^*} = 1$.

Let I be the identity operator on a Banach space X . Recall that X has the *Daugavet property* if the equation

$$\|I + T\| = 1 + \|T\|$$

holds for every rank one operator T on X . Note that if X has the Daugavet property, then the norms of both X and X^* are octahedral [6, Corollary 2.5].

Given two Banach spaces X and Y we will denote by $X \widehat{\otimes}_\varepsilon Y$ the injective, and by $X \widehat{\otimes}_\pi Y$ the projective, tensor product of X and Y . Our main reference for the theory of tensor products of Banach spaces is [24].

A Banach space X has the *diameter two property* (D2P) if every non-empty relatively weakly open subset of B_X has diameter two. X has the D2P if and only if the norm of the dual space is *weakly octahedral*. For the definition of weak octahedrality and its relation to D2P we refer to [14, 20].

According to [16, Definition III.1.1], a Banach space X is said to be an *L -embedded Banach space* if there exists a subspace $Z \subseteq X^{**}$ such that $X^{**} = X \oplus_1 Z$. Note that from the Principle of Local Reflexivity, non-reflexive L -embedded Banach spaces have an octahedral norm.

In Section 4 the theory of almost isometric ideals will play an important role in our results about octahedrality in projective tensor products. Let Z be a subspace of a Banach space X . We say that Z is an *almost isometric ideal* (ai-ideal) in X if X is locally complemented in Z by almost isometries. This means that for each $\varepsilon > 0$ and for each finite-dimensional subspace $E \subseteq X$ there exists a linear operator $T : E \rightarrow Z$ satisfying

- (i) $T(e) = e$ for each $e \in E \cap Z$, and
- (ii) $(1 - \varepsilon)\|e\| \leq \|T(e)\| \leq (1 + \varepsilon)\|e\|$ for each $e \in E$,

i.e. T is a $(1 + \varepsilon)$ isometry fixing the elements of E . If the T 's satisfy only (i) and the right-hand side of (ii) we get the well-known concept of Z being an *ideal* in X [13].

Note that the Principle of Local Reflexivity means that X is an ai-ideal in X^{**} for every Banach space X . Moreover, the Daugavet property, octahedrality and all of the diameter two properties are inherited by ai-ideals (see [1] and [2]).

Let X be a Banach space and let α be a tensor norm. By [24, Proposition 6.4] $X \widehat{\otimes}_\alpha Y$ is a subspace of $X^{**} \widehat{\otimes}_\alpha Y$ for any Banach space Y . A similar argument shows that this result can be generalised to (ai-)ideals, i.e. $Z \widehat{\otimes}_\alpha Y$ is a subspace of $X \widehat{\otimes}_\alpha Y$ for any Banach space Y whenever Z is an ideal in X . In Section 4 we will need the following version of this result:

Proposition 2.1. *Let Z be an (ai-)ideal in X and let Y be a Banach space. Then $Z \widehat{\otimes}_\pi Y$ is a subspace of $X \widehat{\otimes}_\pi Y$.*

With an extra assumption we even get an ai-ideal.

Proposition 2.2. *Let X and Y be Banach spaces. If $L(Y, X^*)$ is norming for $X^{**} \widehat{\otimes}_\pi Y$, then $X \widehat{\otimes}_\pi Y$ is an ai-ideal in $X^{**} \widehat{\otimes}_\pi Y$.*

Proof. We have $(X \widehat{\otimes}_\pi Y)^* = L(X, Y^*) = L(Y, X^*)$ and $(X^{**} \widehat{\otimes}_\pi Y)^* = L(X^{**}, Y^*)$.

Define an operator $\phi : L(Y, X^*) \rightarrow L(X^{**}, Y^*)$ by $\phi(T) := T^*$. We have $\langle \phi(T), u \rangle = \langle T, u \rangle$ for $u \in X \widehat{\otimes}_\pi Y$ and $\|\phi\| \leq 1$, thus ϕ is a Hahn-Banach extension operator. By assumption $\phi(L(Y, X^*))$ is norming, so by [2, Proposition 2.1] we have that $X \widehat{\otimes}_\pi Y$ is an ai-ideal in $X^{**} \widehat{\otimes}_\pi Y$. \square

Recall that a Banach space X has the *metric approximation property* (MAP) if there exists a net (S_α) in $F(X, X)$ such that $S_\alpha x \rightarrow x$ for all $x \in X$. The MAP allows us give examples of spaces where the above proposition applies.

Proposition 2.3. *Let X and Y be Banach spaces. If either X^{**} or Y has the MAP, then $F(Y, X^*) \subset L(Y, X^*)$ is norming for $X^{**} \widehat{\otimes}_\pi Y$. In particular, $X \widehat{\otimes}_\pi Y$ is an ai-ideal in $X^{**} \widehat{\otimes}_\pi Y$.*

Proof. Let $\varepsilon > 0$. Let $u \in X^{**} \widehat{\otimes}_\pi Y$ and choose a representation $u = \sum_{n=1}^\infty x_n^{**} \otimes y_n$ such that $\sum_{n=1}^\infty \|x_n^{**}\| \|y_n\| > \|u\| - \varepsilon$. Choose N such that

$$\sum_{n>N} \|x_n^{**}\| \|y_n\| < \varepsilon.$$

Then for all $T \in L(X^{**}, Y^*)$ with $\|T\| \leq 1$

$$\left| \langle T, u \rangle - \sum_{n=1}^N T x_n^{**}(y_n) \right| \leq \sum_{n>N} \|T x_n^{**}\| \|y_n\| < \varepsilon.$$

Choose $T \in L(X^{**}, Y^*)$ with $\|T\| \leq 1$ such that $\langle T, u \rangle = \|u\|$.

Assume first that Y has the MAP and assume, with no loss of generality, that $\|T x_n^{**}\| \leq 1$ holds for every $n \in \{1, \dots, N\}$. Then there

exists a net $(S_\alpha) \subseteq F(Y, Y)$ such that $\|S_\alpha\| \leq 1$ and $S_\alpha y \rightarrow y$ for all $y \in Y$. Choose α_0 large enough so that $\|S_{\alpha_0} y_n - y_n\| < \varepsilon/N$ for $n \in \{1, \dots, N\}$. Define $T_0 \in F(X^{**}, Y^*)$ by $T_0 := S_{\alpha_0}^* T$. Then

$$\left| \sum_{n=1}^N T x_n^{**}(y_n) - \sum_{n=1}^N T_0 x_n^{**}(y_n) \right| \leq \sum_{n=1}^N \|T x_n^{**}\| \|S_{\alpha_0} y_n - y_n\| < \varepsilon.$$

Similarly, if we assume that X^{**} has the MAP, there exists a net $(S_\alpha) \subseteq F(X^{**}, X^{**})$ such that $\|S_\alpha\| \leq 1$ and $S_\alpha x^{**} \rightarrow x^{**}$ for all $x^{**} \in X^{**}$. Again assume with no loss of generality that $\|T^* y_n\| \leq 1$ holds for every $n \in \{1, \dots, N\}$ and choose α_0 large enough so that $\|S_{\alpha_0} x_n^{**} - x_n^{**}\| < \varepsilon/N$ for $n \in \{1, \dots, N\}$. Define $T_0 \in F(X^{**}, Y^*)$ by $T_0 := T S_{\alpha_0}$. Then

$$\left| \sum_{n=1}^N T x_n^{**}(y_n) - \sum_{n=1}^N T_0 x_n^{**}(y_n) \right| \leq \sum_{n=1}^N \|S_{\alpha_0} x_n^{**} - x_n^{**}\| \|T^* y_n\| < \varepsilon.$$

So far, in both cases we have found $T_0 \in F(X^{**}, Y^*)$ such that

$$|\langle T, u \rangle - \langle T_0, u \rangle| < 3\varepsilon.$$

Next we use [22, Theorem 2.5] to find $T_1 \in F(Y, X^*) = F(X, Y^*)$ such that $\|T_1\| \leq 1 + \varepsilon$ and $T_1^* x_n^{**} = T_0 x_n^{**}$ for $n \in \{1, \dots, N\}$. This implies that

$$\begin{aligned} |\langle T, u \rangle - \langle T_1, u \rangle| &\leq |\langle T, u \rangle - \langle T_0, u \rangle| + |\langle T_0, u \rangle - \langle T_1, u \rangle| \\ &< 3\varepsilon + 2\varepsilon = 5\varepsilon. \end{aligned}$$

Hence we have $\langle T_1, u \rangle > \|u\| - 5\varepsilon$. Since $\varepsilon > 0$ was arbitrary we get that $F(X, Y^*)$ is norming for $X^{**} \widehat{\otimes}_\pi Y$. \square

Related to almost isometric ideals is the notion of finite representability. In Section 3 we shall need a characterisation of when a separable Banach space is finitely representable in ℓ_1 . The following lemmata are probably well-known, but we include their proofs for easy reference.

Lemma 2.4. *Let ν be a σ -additive measure. If a separable Banach space X is finitely representable in $L_p(\nu)$, $1 \leq p < \infty$, then it is isometric to a subspace of $L_p[0, 1]$.*

Proof. By [5, Proposition 11.1.12] X is isometric to a subspace of an ultrapower $Y = (L_p(\nu))_{\mathcal{U}}$ of $L_p(\nu)$ for a nonprincipal ultrafilter \mathcal{U} . But Y is isometric to $L_p(\mu)$ for some measure μ [17, Theorem 3.3]. Since any separable subspace of $L_p(\mu)$ is isometric to a subspace of some separable $L_p(\mu_1)$ [25, Proposition III.A.2], which is isometric to a subspace of $L_p[0, 1]$ [18, pp. 14–15], the lemma follows. \square

Lemma 2.5. *Let X be a separable Banach space. The following are equivalent:*

- (i) X is finitely representable in L_1 .
- (ii) X is finitely representable in ℓ_1 .

(iii) X is isometric to a subspace of L_1 .

(iv) For all $\varepsilon > 0$ there exists a $(1 + \varepsilon)$ isometry from X into L_1 .

These statements are implied by

(v) For all $\varepsilon > 0$ there exists a $(1 + \varepsilon)$ isometry from X into ℓ_1 .

If X is finite-dimensional, then all the statements are equivalent.

Proof. (i) \Rightarrow (ii) since L_1 is finitely representable in ℓ_1 [5, Proposition 11.1.7]. (ii) \Rightarrow (iii) by Lemma 2.4. (iii) \Rightarrow (iv), (iv) \Rightarrow (i) and (v) \Rightarrow (ii) are all trivial.

If X is finite-dimensional, then (iii) \Rightarrow (v) by finite representability of L_1 in ℓ_1 . \square

3. OCTAHEDRALITY IN INJECTIVE TENSOR PRODUCTS

The authors of [15] introduced a new notion of octahedrality. The norm of a Banach space X is *alternatively octahedral* if, for every $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$, there is a $y \in S_X$ such that

$$\max\{\|x_i + y\|, \|x_i - y\|\} > 2 - \varepsilon \quad \text{for all } i \in \{1, \dots, n\}.$$

This norm condition implies that there exist $x_1^*, \dots, x_n^* \in S_{X^*}$ such that

$$|x_i^*(x_i)| > 1 - \varepsilon \quad \text{and} \quad |x_i^*(y)| > 1 - \varepsilon \quad \text{for every } i \in \{1, \dots, n\}.$$

It is known that the norm of X is octahedral if, and only if, for every $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$ there exists $y \in S_X$ such that $\|x_i + y\| > 2 - \varepsilon$ for all $i \in \{1, \dots, n\}$ (see [14, Proposition 2.1]). Consequently, octahedrality implies alternative octahedrality. However, the converse does not hold.

Example. It is not difficult to see that c_0 and ℓ_∞ do not have an octahedral norm. However, the norms of these spaces are alternatively octahedral. To see this consider elements x_1, \dots, x_n of norm one and let i_1, \dots, i_m be distinct indices where these elements (almost) attain their norm. The norm one element $y = e_{i_1} + e_{i_2} + \dots + e_{i_m}$ does the job.

In [21] it is shown that if X and Y are Banach spaces whose norms are octahedral then the norm of $X \widehat{\otimes}_\varepsilon Y$ is also octahedral. The following proposition is similar to [15, Theorem 2.1] and improves [21, Theorem 2.2].

Proposition 3.1. *Let X and Y be Banach spaces and H a subspace of $L(X^*, Y)$ containing $X \otimes Y$ such that every $T \in H$ is weak*-weakly continuous. If the norm of X is alternatively octahedral and the norm of Y is octahedral, then the norm of H is octahedral.*

Proof. Let $T_1, \dots, T_n \in S_H$ and $\varepsilon > 0$. For each $i \in \{1, \dots, n\}$ find $y_i^* \in S_{Y^*}$ such that $\|T_i^* y_i^*\| > 1 - \varepsilon$. Note that $T_i^* y_i^* \in X$ for all $i \in \{1, \dots, n\}$ since H consists of weak*-weakly continuous operators. Since the norm of X is alternatively octahedral there exist $x_1^*, \dots, x_n^* \in S_{X^*}$ and $w \in S_X$ such that $|x_i^*(w)| > 1 - \varepsilon$ and

$$|x_i^*(T_i^* y_i^*)| > \|T_i^* y_i^*\|(1 - \varepsilon) > (1 - \varepsilon)^2$$

holds for every $i \in \{1, \dots, n\}$. We may assume that $x_i^*(T_i^* y_i^*) > 0$ for all $i \in \{1, \dots, n\}$. Define $\gamma_i := \text{sign } x_i^*(w)$.

Let $F = \text{span}\{T_i x_i^* : i \in \{1, \dots, n\}\} \subset Y$. Use octahedrality and [20, Theorem 3.21] to find $z \in S_Y$ and $z_i^* \in Y^*$, $i \in \{1, \dots, n\}$, such that $z_i^*(T_i x_i^*) = y_i^*(T_i x_i^*)$, $z_i^*(z) = \gamma_i$ and $\|z_i^*\| \leq 1 + \varepsilon$ holds for every i .

Define $S := w \otimes z \in X \otimes Y$. We have $S \in S_H$ and, for each $i \in \{1, \dots, n\}$, it follows that

$$\begin{aligned} \|T_i + S\| &\geq \frac{1}{1 + \varepsilon} z_i^*(T_i x_i^* + S x_i^*) = \frac{1}{1 + \varepsilon} (y_i^*(T_i x_i^*) + x_i^*(w) z_i^*(z)) \\ &= \frac{1}{1 + \varepsilon} (y_i^*(T_i x_i^*) + |x_i^*(w)|) > \frac{2 - 3\varepsilon + \varepsilon^2}{1 + \varepsilon} > 2 - 5\varepsilon. \end{aligned}$$

Hence we conclude that the norm of H is octahedral. \square

Throughout the rest of this section we study whether the norm of $X \widehat{\otimes}_\varepsilon Y$ is octahedral when we assume that the norm of only one of the factors is octahedral. For this, we shall begin by giving some positive results for the Banach spaces ℓ_1 and L_1 , which have an octahedral norm.

Theorem 3.2. *Let X be a Banach space. Then:*

- (i) *If X is $(1 + \varepsilon)$ isometric to a subspace of ℓ_1 , then the norm of $L(X, \ell_1)$ is octahedral.*
- (ii) *If X is $(1 + \varepsilon)$ isometric to a subspace of L_1 , then the norm of $L(X, L_1)$ is octahedral.*

Proof. (i). Let $\varepsilon > 0$ and $\psi : X \rightarrow \ell_1$ be a $(1 + \varepsilon)$ isometry. Let $T_1, \dots, T_n \in S_{L(X, \ell_1)}$ and, for every $i \in \{1, \dots, n\}$, pick $x_i \in S_X$ such that $\|T_i(x_i)\| > 1 - \varepsilon$.

Let P_k be the projection on ℓ_1 onto the first k coordinates. Choose $k \in \mathbb{N}$ so that $\|P_k(T_i(x_i)) - T_i(x_i)\| < \varepsilon$ and $\|P_k(\psi(x_i)) - \psi(x_i)\| < \varepsilon$ for every $i \in \{1, \dots, n\}$.

Let $\varphi_k : \ell_1 \rightarrow \ell_1$ be the shift operator defined by

$$\varphi_k(x)(n) := \begin{cases} 0 & \text{if } n \leq k, \\ x(n - k) & \text{if } n > k. \end{cases}$$

Define $S := \varphi_k \circ P_k \circ \psi$. Now, as $P_k(T_i(x_i))$ and $S(x_i)$ have disjoint support, we have that

$$\begin{aligned} \|T_i + S\| &\geq \|P_k T_i(x_i)\| - \varepsilon + \|P_k(\psi((x_i)))\| \\ &\geq \|T_i(x_i)\| + \|\psi(x_i)\| - 3\varepsilon > 2 - 5\varepsilon, \end{aligned}$$

so we are done.

(ii). Define $A := [0, 1]$. Let $T_1, \dots, T_n \in S_{L(X, L_1)}$ and $\varepsilon > 0$. By assumption there exists $x_i \in S_X$ such that $\|T_i(x_i)\| = \int_A |T_i(x_i)| > 1 - \frac{\varepsilon}{2}$ for all $i \in \{1, \dots, n\}$. Pick a closed interval $I \subseteq A$ such that $\int_I |T_i(x_i)| < \frac{\varepsilon}{2}$ holds for each $i \in \{1, \dots, n\}$.

By assumption and Lemma 2.5 there exists a linear isometry $T : X \rightarrow L_1$. Let $\phi : I \rightarrow A$ be an increasing and affine bijection. Define $S_I : L_1 \rightarrow L_1$ by the equation

$$S_I(f) = (f \circ \phi)\phi' \chi_I \quad \text{for all } f \in L_1,$$

where χ_I denotes the characteristic function on the interval I . Note that S_I is a linear isometry because of the change of variable theorem. Indeed

$$\|S_I(f)\| = \int_I |(f \circ \phi)\phi'| = \int_{\phi(I)} |f| = \int_A |f| = \|f\| \text{ for all } f \in L_1.$$

Define $G := S_I \circ T$, which is a linear isometry such that $\text{supp}(G(f)) \subseteq I$ for all $f \in L_1$. Given $i \in \{1, \dots, n\}$, we have

$$\|T_i + G\| \geq \|T_i(x_i) + G(x_i)\| = \int_{A \setminus I} |T_i(x_i)| + \int_I |T_i(x_i) + G(x_i)|.$$

Now

$$\int_{A \setminus I} |T_i(x_i)| = \|T_i(x_i)\| - \int_I |T_i(x_i)| > 1 - \varepsilon.$$

Moreover

$$\int_I |T_i(x_i) + G(x_i)| \geq \int_I |G(x_i)| - |T_i(x_i)| > \int_I |G(x_i)| - \frac{\varepsilon}{2}.$$

Finally note that, as $\text{supp}(G(x_i)) \subseteq I$, we have $\int_I |G(x_i)| = \|G(x_i)\| = \|x_i\| = 1$. Consequently

$$\|T_i + G\| > 2 - 2\varepsilon.$$

As ε was arbitrary we conclude that the norm of $L(X, L_1)$ is octahedral, as desired. \square

From here we can conclude the following result.

Corollary 3.3. *If X is a 2-dimensional Banach space, then the norms of both $\ell_1 \widehat{\otimes}_\varepsilon X = L(c_0, X)$ and $L_1 \widehat{\otimes}_\varepsilon X$ are octahedral.*

Proof. We have that X^* is isometric to a subspace of L_1 [10, Corollary 1.4]. From Lemma 2.5 we see that Theorem 3.2 applies. \square

Note that the above corollary improves [15, Proposition 2.3], where the authors show that the norm of $L(c_0, \ell_p^2)$ is octahedral for every $1 \leq p \leq \infty$. Dualising we get the following result, which improves [21, Proposition 2.10] for two dimensional Banach spaces.

Corollary 3.4. *If X is a 2-dimensional Banach space, then $c_0 \widehat{\otimes}_\pi X$ has the SD2P.*

Next we give more examples of finite-dimensional Banach spaces for which the norm of its injective tensor product with ℓ_1 and L_1 are octahedral.

Proposition 3.5. *Let $n \geq 3$ be a natural number and $2 \leq p \leq \infty$. Then the norms of both $\ell_1 \widehat{\otimes}_\varepsilon \ell_p^n$ and $L_1 \widehat{\otimes}_\varepsilon \ell_p^n$ are octahedral.*

Proof. We know that ℓ_{p^*} is isometric to a subspace of L_1 [5, Theorem 6.4.19] which in turn contains $\ell_{p^*}^n$ isometrically. From Lemma 2.5 we see that Theorem 3.2 applies and shows that the norm of $Y \widehat{\otimes}_\varepsilon \ell_p^n = L(\ell_{p^*}^n, Y)$ is octahedral for $Y = \ell_1$ and $Y = L_1$. \square

In fact, an infinite-dimensional version of the previous result also works.

Proposition 3.6. *Let $2 \leq p < \infty$. Then:*

- (i) *Given a closed subspace H of $L(\ell_{p^*}, \ell_1)$ containing $\ell_p \otimes \ell_1$, then the norm of H is octahedral.*
- (ii) *Given a closed subspace H of $L(\ell_{p^*}, L_1)$ containing $\ell_p \otimes L_1$, then the norm of H is octahedral.*

Proof. (i). We proceed as in Theorem 3.2. Given $T_1, \dots, T_n \in S_H$ and $\varepsilon > 0$ we start by choosing, for every $i \in \{1, \dots, n\}$, an element $x_i \in S_{\ell_{p^*}}$ such that $\|T_i(x_i)\| > 1 - \varepsilon$. Find $m \in \mathbb{N}$ such that $\|P_m(x_i) - x_i\| < \varepsilon$, where P_m is the projection onto the first m coordinates. Since ℓ_{p^*} is finitely representable in ℓ_1 there exists a $(1 + \varepsilon)$ isometry $T : P_m(\ell_{p^*}) \rightarrow \ell_1$. The operator $\psi := T \circ P_m$ is then well-defined and using this ψ we define $S := \varphi_k \circ P_k \circ \psi$ as in the proof of Theorem 3.2. Note that $S \in \ell_p \otimes \ell_1 \subseteq H$ since P_m has finite rank. Similar calculations to the ones in Theorem 3.2 conclude the proof.

The proof of (ii) is similar, but in this case we can use an isometry $T : \ell_{p^*} \rightarrow L_1$. \square

The above results can be seen as sufficient conditions to get octahedrality in injective tensor products spaces. Now we turn to analyse some necessary conditions.

Lemma 3.7. *Let X and Y be Banach spaces and assume that Y^* is uniformly convex. Assume also that there exists a closed subspace H of $L(Y^*, X)$ such that $X \otimes Y \subseteq H$ and that the norm of H is octahedral. Then Y^* is finitely representable in X .*

Proof. Recall that the modulus of uniform convexity of Y^* is defined by

$$\delta(\varepsilon) = \inf \left\{ 1 - \left\| \frac{f+g}{2} \right\| : f, g \in B_{Y^*}, \|f - g\| \geq \varepsilon \right\}.$$

Note that if $f, g \in B_{Y^*}$ satisfy $f(y) > 1 - \delta(\varepsilon)$ and $g(y) > 1 - \delta(\varepsilon)$, for some $y \in S_Y$, then $\|f - g\| < \varepsilon$.

Let $\varepsilon > 0$ and choose $\nu > 0$ so small that $(1 + \nu)(1 - 3\nu)^{-1} < 1 + \varepsilon$. Pick $0 < \eta < \nu/2$ such that $\delta(\eta) < \nu/2$.

Let $F \subseteq Y^*$ be a finite-dimensional subspace. Pick a ν -net $(f_i)_{i=1}^n$ for S_F . Choose $y_i \in S_Y$ such that $f_i(y_i) = 1$.

Let $x \in S_X$. By assumption the norm of H is octahedral, so there exists a $T \in S_H$ such that

$$\|y_i \otimes x + T\| > 2 - \delta(\eta)$$

holds for every $i \in \{1, \dots, n\}$.

We want to show that F is $(1 + \varepsilon)$ isometric to a subspace of X . We have $\|T(f)\| \leq \|f\|$ since T has norm one. For y_i we choose $\varphi_i \in S_{Y^*}$ such that

$$\|\varphi_i(y_i)x + T(\varphi_i)\| > 2 - \delta(\eta).$$

By the triangle inequality $|\varphi_i(y_i)| > 1 - \delta(\eta)$ and $\|T(\varphi_i)\| > 1 - \delta(\eta)$. We may assume that $\varphi_i(y_i) > 1 - \delta(\eta)$. Since $f_i(y_i) = 1$ we get $\|f_i - \varphi_i\| < \eta < \nu/2$. We also get

$$\|T(f_i)\| \geq \|T(\varphi_i)\| - \|T\|\|f_i - \varphi_i\| > 1 - \delta(\eta) - \frac{\nu}{2} > 1 - \nu.$$

From [5, Lemma 11.1.11] we see that T restricted to F is a $(1 + \varepsilon)$ isometry. \square

Using the above lemma we get the following result.

Theorem 3.8. *For every $1 < p < 2$ and every natural number $n \geq 3$ the norms of both $\ell_1 \widehat{\otimes}_\varepsilon \ell_p^n$ and $L_1 \widehat{\otimes}_\varepsilon \ell_p^n$ fail to be octahedral.*

Proof. From [10, Theorem 1.5] we see that ℓ_p^n is not isometric to a subspace of $L_1[0, 1]$. Combining Lemma 2.5 and Lemma 3.7 we get the desired conclusion. \square

Now, if we dualise Theorem 3.8 and use [24, Proposition 5.33], we get the following corollary, which gives a negative answer to an open problem posed in [7, p. 177] as well as in [15].

Corollary 3.9. *Let $2 < p < \infty$ and $n \geq 3$. Then neither $\ell_\infty \widehat{\otimes}_\pi \ell_p^n$ nor $L_\infty \widehat{\otimes}_\pi \ell_p^n$ enjoy the SD2P.*

The above theorem together with Proposition 3.1 and Proposition 3.6 allow us to give the following characterisation of octahedrality when dealing with classic sequence spaces.

Theorem 3.10. *Let $1 \leq p \leq \infty$ and let X be either L_1 or ℓ_1 . Then:*

- (i) If H is a closed subspace of $L(\ell_{p^*}, X)$ which contains $\ell_p \otimes X$, then the norm of H is octahedral if, and only if, $2 \leq p$ or $p = 1$.
- (ii) If H is a closed subspace of $L(\ell_1, X)$ which contains $c_0 \otimes X$, then the norm of H is octahedral.
- (iii) If n is a natural number and H is a closed subspace of $L(\ell_{p^*}^n, X)$ which contains $\ell_p^n \otimes X$, then the norm of H is octahedral if, and only if, either $n \leq 2$ or if $n > 2$ and $2 \leq p$ or $p = 1$.

Theorem 3.8 and Corollary 3.9 also allow us to shed light on a number of results and questions from the literature.

Remark 3.11. In [3, Question (b)] it is asked how diameter two properties are preserved by tensor products. We can now provide a complete answer to this question for the SD2P in the projective case. The SD2P is preserved from both factors, by [7, Corollary 3.6], but not in general from one of them, by Corollary 3.9.

In [21, Question 4.1] it is asked whether octahedrality is preserved by injective tensor products just from one of the factors. Theorem 3.8 gives a negative answer to this question.

Remark 3.12. Note that L_∞ as well as ℓ_∞ have an infinite-dimensional centralizer [16, Example I.3.4.(h)]. From [7, Corollary 3.8] and Corollary 3.9 we see that, given two Banach spaces X and Y , it is not enough to assume that X has an infinite-dimensional centralizer to ensure that $X \widehat{\otimes}_\pi Y$ has the SD2P. But both L_∞ and ℓ_∞ are isometric to $C(K)$ spaces so $L_\infty \widehat{\otimes}_\pi Y$ and $\ell_\infty \widehat{\otimes}_\pi Y$ do have the D2P for any Y by [4, Proposition 4.1].

For some spaces we can say even more. Let $Y = \ell_p$ or $Y = \ell_p^n$ with n a natural number and $1 < p < \infty$. By [24, Proposition 5.33] we have $(\ell_1 \widehat{\otimes}_\varepsilon Y)^* = \ell_\infty \widehat{\otimes}_\pi Y^*$. By [14, Theorem 2.7] we get that the bidual $(\ell_1 \widehat{\otimes}_\varepsilon Y)^{**}$ is weakly octahedral. But, for $1 < p < 2$ and $n \geq 3$, $\ell_1 \widehat{\otimes}_\varepsilon Y$ is not octahedral, by Theorem 3.8, hence $\ell_\infty \widehat{\otimes}_\pi Y^*$ does not even have the w^* -SD2P (see e.g. [6, Theorem 2.1] or [14, Theorem 2.2]).

Remark 3.13. Our results also give natural examples of tensor products failing the Daugavet property.

By Theorem 4.2 and Corollary 4.3 in [19] there exists a two dimensional complex Banach space E such that both $L_1^c \widehat{\otimes}_\varepsilon E$ and $L_\infty^c \widehat{\otimes}_\pi E^*$ fail the Daugavet property. Note that both real and complex L_1 and L_∞ have the Daugavet property.

A Banach space with the Daugavet property has the SD2P and an octahedral norm ([6, Corollary 2.5] and [3, Theorem 4.4]). We can thus improve the above mentioned results of [19] by giving examples of (real) Daugavet spaces such that their tensor product fail to be octahedral or fail to have the SD2P. By Theorem 3.10, $L_1 \widehat{\otimes}_\varepsilon \ell_p^n$ does not have an octahedral norm for $1 < p < 2$ and $n \geq 3$, and by Corollary 3.9, $L_\infty \widehat{\otimes}_\pi \ell_p^n$ does not have the SD2P for $2 < p < \infty$ and $n \geq 3$.

4. OCTAHEDRALITY IN PROJECTIVE TENSOR PRODUCTS

Given two Banach spaces X and Y , no octahedrality assumption is needed on X or Y in order for $X \widehat{\otimes}_\pi Y$ to have an octahedral norm. Indeed, it follows from [16, Corollary III.1.3] and the Principle of Local Reflexivity that the norm of $\ell_2 \widehat{\otimes}_\pi \ell_2$ is octahedral in spite of the fact that ℓ_2 is a Hilbert space. On the other hand, if we assume that one of the factors is finite-dimensional, then the octahedrality of $X \widehat{\otimes}_\pi Y$ forces the other factor to have an octahedral norm.

Proposition 4.1. *Let X and Y be Banach spaces. Assume that Y is finite-dimensional and that $X \widehat{\otimes}_\pi Y$ has an octahedral norm. Then X has an octahedral norm.*

Proof. Pick $x_1, \dots, x_n \in S_X$ and $\varepsilon > 0$. Since $X \widehat{\otimes}_\pi Y$ has an octahedral norm and Y is finite-dimensional we can find by [23, Proposition 3.2] $u \otimes v \in S_X \otimes S_Y$ such that

$$\|x_i \otimes y + u \otimes v\| \geq 2 - \varepsilon$$

holds for all $y \in Y$ and $i \in \{1, \dots, n\}$. Now, given $i \in \{1, \dots, n\}$, we have

$$2 - \varepsilon \leq \|x_i \otimes v + u \otimes v\| = \|x_i + u\| \|v\| = \|x_i + u\|.$$

Hence, X has an octahedral norm, as desired. \square

Lemma 4.2. *Let X and Z be Banach spaces. If Z is an ai-ideal in X and, for every $z_1, \dots, z_n \in S_Z$ there exists $v \in S_X$ such that*

$$\|z_i + v\| = \|z_i\| + \|v\| \quad \text{for all } i \in \{1, \dots, n\},$$

then the norm of Z is octahedral.

Proof. Let $z_1, \dots, z_n \in S_Z$, $\varepsilon > 0$ and v as in the hypothesis of the lemma. Define $E := \text{span}\{v, z_1, \dots, z_n\}$. Find $T : E \rightarrow Z$ such that $T(e) = e$ for all $e \in E \cap Z$ and

$$(1 - \varepsilon)\|e\| \leq \|T(e)\| \leq (1 + \varepsilon)\|e\| \quad \text{for all } e \in E.$$

Let $q = \frac{T(v)}{\|T(v)\|} \in S_Z$. We have

$$\begin{aligned} \|z_i + q\| &\geq \|z_i + T(v)\| - \varepsilon = \|T(z_i + v)\| - \varepsilon \\ &\geq (1 - \varepsilon)\|z_i + v\| - \varepsilon = 2 - 3\varepsilon, \end{aligned}$$

which means that the norm of Z is octahedral. \square

The following theorem provides a partial positive answer to [21, Question 4.4], where it is asked whether octahedrality is preserved by taking projective tensor products from one of the factors.

Theorem 4.3. *Let X be a non-reflexive L -embedded space and Y be a Banach space. If either X^{**} or Y has the MAP then $X \widehat{\otimes}_\pi Y$ has an octahedral norm.*

Proof. Since X is a non-reflexive L -embedded Banach space then $X^{**} = X \oplus_1 Z$ for some non-zero subspace Z of X^{**} , hence $X^{***} = X^* \oplus_\infty Z^*$.

Let $u \in S_Z$, $y \in S_X$, and $y^* \in S_{Y^*}$ such that $y^*(y) = 1$ and define $v = u \otimes y$. Denote by $X_u = \text{span}\{X, u\} = X \oplus_1 \mathbb{R}$. By the triangle inequality $\|z + v\| \leq \|z\| + \|v\|$ in $X_u \widehat{\otimes}_\pi Y$ for all $z \in X \widehat{\otimes}_\pi Y$. First we will show that we in fact have equality here.

To this aim let $z \in X \widehat{\otimes}_\pi Y$ and pick $T \in S_{L(X, Y^*)}$ such that $\langle T, z \rangle = \|z\|$. Define $\bar{T} : X_u \rightarrow Y^*$ by the equation

$$\bar{T}(x + \lambda u) = T(x) + \lambda y^*.$$

We claim that $\|\bar{T}\| \leq 1$. Indeed, given an arbitrary $x + \lambda u \in X_u$, one has

$$\|\bar{T}(x + \lambda u)\| = \|T(x) + \lambda y^*\| \leq \|T(x)\| + |\lambda| \leq \|x\| + |\lambda| = \|x + \lambda u\|.$$

Consequently, it follows that

$$\|z + v\| \geq \langle \bar{T}, z + v \rangle = \langle T, z \rangle + \langle \bar{T}, v \rangle = \|z\| + y^*(y) = \|z\| + 1 = \|z\| + \|v\|.$$

We have that X_u^* is isometric to $X^* \oplus_\infty \mathbb{R}$, which is an isometric subspace of $X^* \oplus_\infty Z^* = X^{***}$. This implies the existence of a Hahn-Banach operator $\varphi : X_u^* \rightarrow X^{***}$ hence X_u is an ideal in X^{**} [11, Théorème 2.14]. By Proposition 2.1 we conclude that $X_u \widehat{\otimes}_\pi Y$ is an isometric subspace of $X^{**} \widehat{\otimes}_\pi Y$, so

$$\|z + v\|_{X^{**} \widehat{\otimes}_\pi Y} = 1 + \|z\|_{X^{**} \widehat{\otimes}_\pi Y}$$

holds for every $z \in X \widehat{\otimes}_\pi Y$. By Proposition 2.3, $X \widehat{\otimes}_\pi Y$ is an ai-ideal in $X^{**} \widehat{\otimes}_\pi Y$, so Lemma 4.2 finishes the proof. \square

For general Banach spaces X and Y the question of whether $X \widehat{\otimes}_\pi Y$ has an octahedral norm whenever X and/or Y do remains open. We note that it is enough to consider separable Banach spaces to answer this question. This follows by using [1, Theorem 1.5] and Proposition 2.1.

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